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Ising Machine Based on Coupled Spin Torque Oscillators

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Abstract

The Abstract highlights briefly the motivation and identifies the problem and the possible solution that will be examined.

The Ising machine is a recently proposed non-von Neumann computing architecture that can solve combinatorial optimization problems which are unsolvable on existing digital hardware. Many optimization problems can be mapped to an equivalent Ising Hamiltonian and solved using a network of coupled oscillators. However, a scalable hardware implementation of the Ising machine has yet to be realized. Ising machines based on electrical, optical, and magnetic oscillators have all been proposed, but have significant limitations in operating speed, size, and adaptability to different problems. The spin torque oscillator is an emerging computing technology consisting of a nanomagnet driven into auto-oscillations by spin polarized current. Due to its sub-micrometer size, high operating frequency in the GHz range, and high tunability, the spin torque oscillator is a favorable technology to be implemented in a scalable Ising machine. The proposed thesis will analyze the capability of electrically coupled spin torque oscillators to realize a functional Ising machine through physics-based device modeling and analog circuit simulations.

Contents

In the introduction, the author details the motivation for the work and outlines the problem and potential solution. This section also explains and introduces the basic concepts that will be used.

1 Introduction

As Moore's Law scaling comes to an end, innovative devices and architectures are required to enable further improvements in computing efficiency. One such alternative computing paradigm is the Ising machine. This is a highly specialized computing system designed to solve a single problem efficiently, namely finding the energy minimum of the Ising Hamiltonian. Many combinatorial optimization problems can be mapped to an equivalent Ising Hamiltonian and are therefore solvable using an Ising machine. These problems are unsolvable in their full form on existing digital hardware, so realizing a functional Ising machine will significantly enhance computing capabilities.

While the theory behind the Ising machine is elegant, realizing a scalable hardware implementation of this architecture has proven difficult. The solution of the Ising Hamiltonian is found using a network of coupled oscillators, and as the network increases in size, problems arise due to the scaling of the solution time or an inability to synchronize a large number of oscillators. Compared to previous approaches, spin torque oscillators show several promising characteristics for the implementation of a scalable Ising machine, including their sub-micrometer size and high operating frequency in the GHz range. This document proposes a simulation study that will demonstrate the ability of electrically coupled spin torque oscillators to realize a scalable Ising machine.

Benefit of the potential solution

1.1 Ising Hamiltonian

Mathematical expression

The Ising model is a statistical mechanical model that describes domain formation in ferromagnets (1). The Ising Hamiltonian is given by

$$H = -\sum_{1 \le i \le j \le n} J_{ij} s_i s_j - \sum_{i=1}^{n} h_i s_i \tag{1}$$

where s_i are discrete variables representing atomic spins in the ferromagnetic material lattice, of which there are n. Spins may be oriented in the "up" or "down" configuration, taking binary values ± 1. J_{ij} are coupling coefficients between neighboring spins that describe both the polarity and the magnitude of the interaction, and h_i represents an external magnetic field applied to the material.

The Ising model is commonly simplified by neglecting h_i terms, becoming

$$H = -\sum_{i,j,i<j} J_{ij} s_i s_j \tag{2}$$

Many combinatorial optimization problems can be mapped to the Ising Hamiltonian by adjusting the coupling coefficients J_{ij}, including all 21 on Karp's list of NP-Complete problems (2). Due to large solution time scaling with problem size, these optimization problems cannot be solved completely on today's computers. Therefore, realizing a functional Ising machine that can solve this type of problem efficiently would mark a significant advancement in computing capabilities.

1.2 Oscillator-Based Ising Machine

Relates oscillators to Ising machine mathematical definition

Hardware proposed to solve the Ising problem consists of coupled self-sustaining nonlinear oscillators with spins s_i encoded in the oscillator phases and coefficients J_{ij} implemented by the coupling strength between oscillators i and j. Oscillators may be coupled electrically, in which case the microwave signal generated by each oscillator is fed back into each other oscillator through a resistive coupling network with coupling coefficients J_{ij} proportional to the conductances linking the oscillators.

When the oscillators are coupled, they exhibit nonlinear effects such as injection locking, or the synchronization of the oscillator frequency and phase to an external signal. This enables the coupled network of oscillators to find a solution of the Ising model by stably locking the oscillators in phase or with a finite phase difference relative to each other. Because spins in the Ising model take binary values ± 1, a weak second-harmonic locking signal is often injected into the coupled oscillator network to ensure the oscillators lock with precisely 0 or π phase difference, encoding spins ± 1.

The dynamics of the coupled oscillator network are governed by a global Lyapunov function that is functionally equivalent to the Ising model energy minimization problem, so the stable phase configuration of the oscillator network corresponds to the ground state of the Ising Hamiltonian (3) (4). When coupling coefficients J_{ij} are chosen to represent a combinatorial optimization problem of interest, this ground state is the optimal solution.

This scenario describes an idealized picture of the operation of the oscillator Ising machine. In reality, there are considerable difficulties in avoiding local energy minima of the Lyapunov function, especially in a large array of oscillators. In this case, a more complex annealing scheme or noise injection is necessary to ensure the coupled oscillator network reaches the global energy minimum.

1.3 Spin Torque Oscillator

Introduces the oscillator that will be used (STO) and explains it

A nanomagnet excited by spin-polarized current can be used to realize a self-sustaining nonlinear oscillator, called a spin torque oscillator (STO). Consider a magnetic tunnel junction (MTJ) consisting of two ferromagnetic layers separated by an insulating tunnel barrier (Figure 1). The MTJ nanopillar sits on a heavy metal strip. A heavy metal with strong spin-orbit interaction is chosen such that a charge current I_{HM} through the strip creates a spin-polarized current flowing along the normal direction of the metal by the spin Hall effect. The spin current at the heavy metal-MTJ free layer (FL) interface induces a coherent precession of the magnetic moments in the FL about the static field H_{ext}.

Meanwhile, a DC charge current I_{MTJ} is passed through the magnetically polarized layer (PL) of the MTJ. Due to the tunnel magnetoresistance effect, the MTJ exhibits a sinusoidally varying resistance due to the precessing FL moment. I_{MTJ} produces a corresponding microwave voltage signal that can be read out from the MTJ (5). The frequency, linewidth, and power of the spin torque oscillator output signal depend on the input DC and RF power to the oscillator.

The spin torque oscillator's properties are highly tunable based on the geometry, materials, and applied magnetic field. The oscillator can be designed to operate at frequencies well into the GHz range with sub-micron dimensions, making it attractive for use in dense,

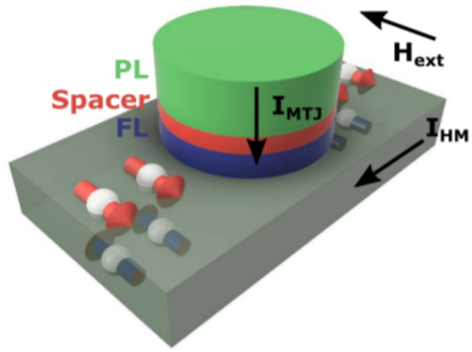

Figure 1: Schematic of MTJ-based spin Hall oscillator. (6)

high-speed computing arrays.

2 Related Work

This section highlights previous attempts and work towards solving the problem and their limitations

2.1 Prototypical Ising Machine

Previous works have demonstrated the feasibility of implementing an Ising machine on a small scale using coupled nonlinear oscillators. Ising machines have been experimentally and theoretically proposed using electronic, optical, and magnetic oscillators.

2.1.1 Electronic

Prototypical Ising machines have been previously realized using self-sustaining LC oscillators (7) (8). Most recently, the physical implementation of an Ising machine up to 240 nodes was demonstrated (9).

A significant limitation of the electronic Ising machine is the operating frequency, as the proposed LC oscillator-based machines operate at a peak frequency of 5 MHz. The time required to find the ground state of the Ising Hamiltonian inversely scales with the operating frequency of the machine. Spin torque oscillators can be easily designed to operate well into the GHz frequency range, making them a more attractive candidate for a scalable Ising machine than electronic oscillators.

Superiority of the author's proposed solution

2.1.2 Optical

A second class of Ising machines based on optical parametric oscillators have been proposed up to 2000 nodes (10), (11). While optical implementations demonstrate high accuracy and efficient solution times for the Ising problem, they require the use of optical fibers up to 1 km long, and so are not practical for dense oscillator networks that will be necessary to solve large-scale optimization problems.

2.1.3 Magnetic

A magnetic oscillator-based Ising machine using magnetodipolar rather than electrical coupling has also been proposed (12). While magnetic coupling mechanisms exhibit significantly lower loss than electrical coupling due to the lack of Ohmic conduction, the tunability of dipolar coupling is extremely limited. Modulating the dipolar coupling strength requires physically moving the oscillators relative to each other, which does not suit a scalable implementation of the magnetic Ising machine. Alternatively, electrical coupling of magnetic oscillators combines the benefits of using magnetic oscillators with the high tunability of resistive coupling.

2.2 Computing with Spin Torque Oscillators

The most significant obstacle preventing spin torque oscillators from being used in real computing applications is the difficulty of device fabrication based on their small size and sensitivity to fabrication variations. As fabrication technologies are improved, using spin torque oscillators for computing is becoming a more realizable goal. A number of recent works have successfully demonstrated the use of spin torque oscillators in computing applications.

Why the author's proposed solution has become more realizable

A single spin torque oscillator has been used for tasks such as waveform recognition by synchronizing the oscillator to external signals corresponding to sine and square waves and reading out the oscillator amplitude, phase, and frequency (13). By tuning the oscillator operating range to take advantage of nonlinearities, accuracies over 90% were obtained. Further developments have led to using a spin torque oscillator to recognize spoken digits by synchronizing the oscillator to an audio signal then matching digits to the spin torque oscillator microwave output signal (14).

The more challenging task of using coupled spin torque oscillators for computing has recently been achieved by synchronizing oscillators through their generated microwave signals. Spoken vowel recognition rates can be improved over single-oscillator rates by the collective synchronization dynamics of serially connected spin torque oscillators (15). Fully-connected networks of spin torque oscillators with resistive coupling can similarly lead to higher recognition rates when applied to pattern matching problems (16).

The recent realizations of high-accuracy pattern recognition schemes using individual and coupled spin torque oscillators demonstrate the increasing readiness of spin torque oscillators to be applied in real computing applications, including the Ising machine.

The use of STO in Ising machine, introduces the link between previous work and the proposed work

3 Proposed Work

This section details the proposed work, identifies the steps and methods that will be followed, and the available and expected results. **Also, note that this section is almost 70% of the whole proposal.**

For my thesis work, I propose to perform a simulation study to analyze the feasibility of using spin torque oscillators to realize a scalable Ising machine. Beginning at the device level and working up to the circuit level, I will study how the nonlinear oscillator characteristics affect inter-oscillator coupling dynamics and interactions with electronic circuit elements to extract relevant parameter considerations in designing a spin torque oscillator network for the Ising machine.

Summary of the entire proposal in one sentence

My results will outline important considerations for using spin torque oscillators in large-scale, circuit-level computing systems. This will enable future work in using spin torque oscillators not only for the Ising machine, but also for other large-scale computing applications which rely on coupled oscillators or interfacing magnetic devices with electronic circuit components.

3.1 Methods

A variety of simulation tools will be required to model the different components of the Ising machine. The simulation tools range from those for physics-based modeling to analog circuit analysis, and will be used together to accurately analyze the magnetic and electronic components of the Ising machine, as well as their interactions.

3.1.1 Micromagnetic Simulation

The micromagnetic simulation software mumax^3 will be used for accurate physical modeling of the spin torque oscillator devices (17). The mumax^3 software is a GPU-accelerated micromagnetic simulation software that allows for efficient simulation of magnetic materials and devices at a sub-micron scale. In micromagnetic simulation, the material of interest is split into discretization cells that are each assumed to have a uniform magnetic moment. Then, the Landau-Lifshitz-Gilbert equation with Slonczewski spin transfer torque term (LLG-S equation) is solved on each cell based on its interactions with neighboring cells and external applied fields and currents to accurately model the system's magnetic dynamics.

The (decoupled) LLG-S equation is given by

$$\frac{d\mathbf{m}}{dt} = -\frac{\gamma}{1+\alpha^2}[\mathbf{m} \times \mathbf{B}_{eff} + (\mathbf{m} \times (\mathbf{m} \times \mathbf{I}_s)) + \alpha(\mathbf{m} \times (\mathbf{m} \times \mathbf{B}_{eff})) - \alpha(\mathbf{m} \times \mathbf{I}_s)] \tag{3}$$

where γ is the gyromagnetic ratio of a free electron and α is the Gilbert damping parameter which is characteristic of the material. Vectors $\mathbf{m}$, $\mathbf{B}_{eff}$, and $\mathbf{I}_s$ represent the magnetic moment (normalized), effective field, and spin current, respectively.

First, a single spin torque oscillator will be simulated using mumax^3 to characterize its nonlinear microwave properties. The nonlinearity of the oscillator will be characterized in detail as a function of input DC and RF power. This information will be used later in constructing accurate circuit-level device models of the oscillator that encapsulate the nonlinear properties and behavior, including phenomenon such as injection locking.

Next, mumax^3 will be used to evaluate the dynamics of electrically coupled spin torque oscillators. The simulation software is not capable of simulating several coupled devices simultaneously by default, but the code is open-source and can therefore be modified for this purpose. A preliminary analysis of a small-scale Ising machine consisting of four coupled oscillators will be carried out in this manner. This analysis will not include any considerations about electronic components, but will serve as a preliminary study of the coupled oscillator phase dynamics and capability of solving the Ising problem. These preliminary coupling results will also be used to verify that the more abstract circuit-level coupling simulation results accurately model the physics of the coupled system.

3.1.2 Numerical Simulation

A secondary method of simulating magnetic devices relies on the macrospin approximation for the LLG-S equation and can be solved numerically in Matlab. The macrospin approximation assumes a uniform magnetic moment throughout the entire modeled material, which is an acceptable approximation for spin torque oscillators with sufficiently small dimensions. While micromagnetic simulations may take multiple hours to run, a similar simulation can be carried using the macrospin approximation in less than a few minutes. Therefore, numerical simulations can supplement the results of micromagnetic simulations in analyzing singular or coupled spin torque oscillators.

3.1.3 Analog Circuit Modeling

The commercial analog circuit simulation software HSPICE will be used to analyze the spin torque oscillator Ising machine from a high-level point of view. This software is flexible and allows for the integration of custom devices and accurate CAD models of electronic components provided by semiconductor companies. Therefore, HSPICE simulations will provide important insights on device- and circuit-level design constraints based on the interplay of the magnetic oscillators and electronic circuit elements. Circuit-level simulations can also be used to model a larger network of oscillators due to the higher level of abstraction, so HSPICE simulations will be used to investigate the scalability of the spin torque oscillator Ising machine and predict its performance using a network with up to 1000 oscillators.

3.1.4 Oscillator Device Macromodeling

An abstract device model of the spin torque oscillator will be designed for the circuit simulation using the analog circuit modeling language Verilog-A. A macromodeling approach will be taken to emulate the observable behavior of the oscillator without requiring the simulator to perform computationally expensive physical calculations.

The first component of the device macromodel is the oscillator nonlinearity. The nonlinear dependence of generated frequency, linewidth, and output power on input DC current will first be determined by mumax^3. There is also a dependence of these parameters on input RF current, but we neglect this dependence by assuming the RF perturbation is weak compared to the DC component. Additionally, the oscillator DC operating point is tuned such that RF injection does not produce a power modulation. In this case, the nonlinear oscillator frequency, linewidth, and power are implemented in the Verilog-A device model as a lookup table indexed by input DC current.

Injection locking is the second effect to be modeled by the oscillator macromodel. Because we assume the RF injection is weak compared to the DC driving current, the nonlinear oscillator can be linearized about its DC operating point and treated similarly to a linear oscillator under perturbation. In this case, the phase dynamics are governed by the well-known Adler equation for a linear oscillator phase under perturbation (18)

$$\frac{d}{dt}\phi(t) = \omega_0 - \omega^* + \omega_0 A \sin(\phi(t) - \phi_{in}(t)) \tag{4}$$

where ω_0, $\phi(t)$ are the frequency and phase of the oscillator, and ω^*, $\phi_{in}(t)$ are the frequency and phase of the perturbation. An equivalent equation for the phase shift $\alpha(t)$ in units of time is

$$\frac{d}{dt}\alpha(t) = p(t + \alpha(t))b(t) \tag{5}$$

where $p(t + \alpha(t))$ is the impulse sensitivity function of the oscillator and $b(t)$ is the perturbation.

The impulse sensitivity function represents the sensitivity of the oscillator phase to external perturbation at a given time in the oscillator period (19). It is a characteristic of the oscillator and independent of the external perturbation (assuming the perturbation is small). As is suggested by Eq. 5, the impulse sensitivity function is significant in determining the phase dynamics of the oscillator.

In the case of the spin torque oscillator, there is a clear angular dependence of the impulse sensitivity function based on the relative orientations of the moments in the exciting (spin Hall) and precessing (free) layers given by the LLG-S equation (Eq. 3). When the microwave output voltage of the spin torque oscillator is proportional to sin(), the impulse sensitivity function will be cos(). The amplitude of the impulse sensitivity function p and perturbation b determine the speed of the phase dynamics, and can be determined analytically or numerically from the LLG-S equation (20).

Finally, the output signal of the oscillator is incorporated into the macromodel to interface with the rest of the circuit. As described in Section 1.3, the tunnel magnetoresistance and DC current flowing through the MTJ produce a microwave output voltage from the oscillator. The resistance of the MTJ varies sinusoidally with the precessing free layer magnetic moment, with maximum resistance occurring when the polarized and free layer moments are most antiparallel and mimimum resistance when the polarized and free layer moments are most parallel. The oscillator output signal will be numerically calculated based on the MTJ resistance and current values, as well as the nonlinear generated frequency, linewidth, and power.

3.2 Preliminary Results

Results found so far using each of the previously mentioned methods are presented and future steps are discussed

3.2.1 Device-Level Simulation

I have performed preliminary simulations of single spin torque oscillator devices using mumax^3 and macrospin numerical modeling. These simulations provide a general overview of the nonlinear characteristics of the oscillator in the chosen geometry that can serve as a starting point for developing a macromodel for the spin torque oscillator.

The 3-terminal spin Hall oscillator geometry was chosen due to high tunability of the oscillator characteristics and ease of microwave signal readout that enables integration into an electronic circuit (5). A typical size and material composition were chosen (50 nm diameter, Permalloy free and polarized layers) as a starting point to investigate the nonlinear oscillator characteristics. The generated frequency, linewidth, and power of the spin torque oscillator were determined as a function of DC input current. In addition, injection locking under first- and second-harmonic injected currents were analyzed, and the locking range and locked frequency, linewidth, and power were determined as a function of RF and DC input currents. Knowing these nonlinear dependences is sufficient to begin macromodeling of the spin torque oscillator device for circuit-level simulation.

When the oscillator parameters, such as size, material, and applied field, are changed later to fit circuit-level constraints and optimize the Ising machine performance, the fundamental operation of the device will remain the same. Therefore, any models developed from these preliminary mumax^3 and macrospin results will be applicable later by changing some numerical parameters.

3.2.2 Electrical Coupling in Micromagnetic Simulation

I adapted the mumax^3 source code to model up to four electrically coupled spin torque oscillators. The additions to the source code abstractly model a resistive coupling network, where microwave output signals from each oscillator are fed to each other oscillator in the network with appropriate weighting by specified conductance values. The four oscillators are analyzed simultaneously by specifying a large simulation geometry where the oscillators are spaced far enough apart such that dipolar field effects are negligible compared to the electrical coupling strength.

Preliminary coupling simulations were conducted to demonstrate the functionality of a four-node spin torque oscillator Ising machine under various coupling coefficients and strengths. The oscillator phases were found to settle in the correct ground state configuration in cases of both ferromagnetic $J_{ij} > 0$ and antiferromagnetic $J_{ij} < 0$ coupling. The solution time was found to scale inversely with oscillator frequency and coupling strength, and the addition of a second-harmonic injection locking signal slightly reduced the solution time by pushing the oscillator phase differences toward 0 and π radians. These observations all agree qualitatively with previous Ising machine implementations.

The benefits of using spin torque oscillators for the Ising machine were also visible in the mumax^3 coupling simulations. First, due to the high operating frequency, the ground state solution time was on the order of nanoseconds, whereas previous LC-oscillator Ising machines observed solution times on the order of microseconds (8) (7). Due to neglecting circuit-level nonidealities such as delay and noise, the mumax^3 solution time is highly optimistic but still suggests that spin torque oscillators can be used to realize an Ising machine with fast operation.

The high tunability of the oscillator also enabled the optimization of the oscillator's DC operating point, which modulates the strength of the oscillator response to an external RF signal. Close to threshold, the oscillator is very sensitive to external perturbation and the operating point is strongly modulated by the RF injection. Slightly further from the threshold current, the oscillator is still sensitive to external perturbation but the operating point is more stable, which is a more optimal state for the coupled oscillator network. The high tunability of the oscillator provides flexibility that will aid in the circuit-level design of the coupled oscillator system.

3.2.3 Physics-Based Device Modeling

Based on preliminary studies of the oscillator nonlinearity and injection locking using mumax^3 and macrospin, I have begun developing a physics-based macromodel of the spin torque oscillator device.

As a first step, the oscillator device model is implemented in Verilog-A at a single operating point (nonlinearities are not accounted for, as the DC current is kept at one value). At this operating point, which has also been analyzed using mumax^3 and macrospin numerical simulations, the phase dynamics are analyzed using the linearized phase model given by Eq. 5.

The main analysis of the linearized phase model has to do with the oscillator impulse sensitivity function. I numerically calculated this function using numerical methods based on the LLG-S equation (20). The numerical method involves computing the state-transition matrix of the oscillator, which describes the time-evolution of the oscillator in state space (with states denoted by different spatial configurations of the magnetic moments in the free layer). From here, an analysis of the eigenvectors and eigenvalues of the state-transition matrix yields the impulse sensitivity function of the auto-oscillatory mode. The extracted impulse sensitivity function qualitatively agrees with the LLG-S equation.

Quantitatively, the Adler equation (Eq. 4) has been analyzed in related works to describe the injection locking of a spin torque oscillator (21). The amplitude A in the Adler equation can be calculated based on the torque in the LLG-S equation corresponding to the RF excitation. This coupling amplitude can then be carried through the mathematical transformation from the Adler equation to the phase equation in units of time (Eq. 5) to find an equivalent amplitude for the macromodel phase equation. The time constant of the phase dynamics based on the coupling amplitude in the macromodel phase equation quantitatively agrees with the mumax^3 result.

The quantitative agreement of the linearized phase equation with the mumax^3 coupling results confirms the validity of the phase macromodeling approach. In the proposed work, this preliminary device macromodel is to be expanded to include nonlinearities, and will then be integrated into full circuit-level simulations of the spin torque oscillator Ising machine.

3.3 Timeline

Concrete deliverables are presented month by month, containing the timeline for both previous work and future work

Sept-Oct. 2020	Preliminary study of spin torque oscillator devices and coupled dynamics using mumax^3
Nov-Dec. 2020	Develop physics-based spin torque oscillator device model in Verilog-A
	Calibrate preliminary HSPICE results to previous mumax^3 simulations
Jan-Feb. 2021	Integrate circuit-accurate CAD models for electronic components into HSpice simulation
	Investigate how magnetic coupling dynamics are affected by circuit delays
Mar-Apr. 2021	Expand size of HSPICE simulated system from a few oscillators up to a few hundred - 1000
May 2021	Write thesis

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All citations are presented in standardized style

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